Descriptive complexity of linear algebra

Bjarki Holm
Overview

Study **definability** of natural problems in linear algebra and **expressiveness** of logics with algebraic operators.

- Background & motivation
- Descriptive complexity of problems in linear algebra
- Logics with matrix-rank operators
- Pebble games for rank logics & the Weisfeiler-Lehman method
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A logic for \textbf{NP}

**ESO — Existential second-order logic**
Second-order variables existentially quantified, followed by a first-order formula:

$$
\exists R_1, \ldots, R_k \cdot \varphi(R_1, \ldots, R_k)
$$
A logic for **NP**

A decision problem is in **NP** if and only if it can be defined in ESO.

Fagin (1974)

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“guess”

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“guess” \quad “verify”

Is there a logic for $\textbf{PTIME}$?
A logic for PTIME?
Fixed-point logic captures PTIME on ordered structures

$\text{FP}$ is first-order logic with an inflationary fixed-point operator.

A property $P$ of ordered structures can be decided in PTIME if and only if $P$ can be defined by a sentence of FP.

Immerman-Vardi (1982)
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\textbf{FP} is first-order logic with an \textit{inflationary fixed-point} operator.

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\textbf{Immerman-Vardi (1982)}

\textit{Ordered structure:} Vocabulary contains a binary symbol “\( \leq \)” interpreted as a total ordering of the vertices.
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- On unordered structures, FP cannot even express if a graph has an even or odd number of vertices.
- *Fixed-point logic with counting* (FPC) is FP together with terms that count the number of solutions to formulas.
FPC captures $\text{PTIME}$ on...
FPC captures PTIME on...
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Ordered structures—1982

Trees—1986

PTIME

FPC

FP

FO
FPC captures PTIME on...

- Trees—1986
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PTIME

FPC
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FPC captures PTIME on...

- Ordered structures—1982
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- Planar graphs—1998
- Graphs of bounded treewidth—1999

Diagram showing relationships between FPC, FP, and FO with respect to PTIME.
FPC captures PTIME on...

- Minor-closed classes of graphs—2010
- Graphs of bounded treewidth—1999
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PTIME

FPC

FP

FO
FPC captures PTIME on...

- Minor-closed classes of graphs—2010
- Graphs of bounded treewidth—1999
- Planar graphs—1998
- “Almost all” graphs—1996
- Trees—1986
- Ordered structures—1982
FPC captures PTIME on... all graphs?
Proving non-definability in FPC

$C^k$ — first-order logic with variables $x_1, ..., x_k$ and counting quantifiers of the form $\exists \geq^i x . \varphi$
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Proving non-definability in FPC

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$G$ and $H$ agree on all sentences of $C^k$ iff Duplicator has a winning strategy in the $k$-pebble bijection game on $G$ and $H$
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For each $k$, exhibit a pair of graphs $G_k$ and $H_k$ for which
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• Duplicator wins the $k$-pebble game on $G_k$ and $H_k$. 
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Facts
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- For each \( k \), we can decide the winner of the \( k \)-pebble game in polynomial time.
Proving non-definability in FPC

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Facts

• For each $k$, we can decide the winner of the $k$-pebble game in polynomial time.

• Close connection with a family of algorithms for graph isomorphism: Weisfeiler-Lehman method.
Non-definability result for FPC

There is a polynomial-time decidable property of finite graphs that is not definable in FPC.

Cai, Fürer and Immerman (1992)
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Corollary

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FPC does not capture PTIME on

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- graphs of bounded colour-class size (not even degree 3)
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Still, the CFI query is hardly a natural graph property...

More recently: See which problems in linear algebra can be expressed in FPC
Descriptive complexity of problems in linear algebra
The usual notion of a matrix

\[ A = (a_{ij}) \quad \text{— an } m\text{-by-}n \text{ rectangular array of elements} \]
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Recall: Over ordered structures FP (and hence FPC) can define all polynomial-time properties.
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**Recall:** Over ordered structures FP (and hence FPC) can define *all* polynomial-time properties.

- rows and columns ordered
- all PTIME matrix properties can be defined in FP
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Many natural matrix properties \textit{invariant under permutation} of rows and columns
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\[ \text{rows and columns ordered} \rightarrow \text{all PTIME matrix properties can be defined in FP} \]

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Recall: Over ordered structures FP (and hence FPC) can define all polynomial-time properties.

Many natural matrix properties **invariant under permutation** of rows and columns

(rank, determinant, etc.)
Unordered matrices

$I, J$ — finite and non-empty sets
$D$ — a group, a ring or a field
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$I, J$ — finite and non-empty sets

$D$ — a group, a ring or a field

$$A : I \times J \to D$$
Unordered matrices

$I, J$ — finite and non-empty sets
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$A : I \times J \to D$  
“an $I$-by-$J$ matrix over $D$”
Unordered systems of linear equations

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Unordered systems of linear equations

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$A x = b$
Unordered systems of linear equations

\[ A x = b \]
Unordered systems of linear equations

As a relational structure over a fixed domain $D$:

$$Ax = b$$
Unordered systems of linear equations

As a relational structure over a fixed domain $D$:

$$\mathcal{S} = (I, J; (A_d)_{d \in D}, (b_d)_{d \in D}) \quad \text{where} \quad A_d \subseteq I \times J \quad \text{and} \quad b_d \subseteq I$$

$$A \mathbf{x} = \mathbf{b}$$
Unordered systems of linear equations

As a relational structure over a fixed domain $D$:

$$G = (I, J; (A_d)_{d \in D}, (b_d)_{d \in D}) \quad \text{where} \quad A_d \subseteq I \times J \quad \text{and} \quad b_d \subseteq I$$

$$A \times x = b$$
Unordered systems of linear equations

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\[
\begin{bmatrix}
A_0
\end{bmatrix}
\begin{bmatrix}
I
\end{bmatrix}
\begin{bmatrix}
t
\end{bmatrix}
\begin{bmatrix}
J
\end{bmatrix}
= 
\begin{bmatrix}
I
\end{bmatrix}
\]

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$$A_1$$

$$I$$

$$J$$

$$Ax = b$$

In this talk: Focus on $I = J$
FPC — more non-definability results

Solvability of systems of linear equations over any fixed finite Abelian group is not definable in FPC.

Atserias, Bulatov and Dawar (2007)
FPC — more non-definability results

Corollary

Solvability of systems of linear equations over any fixed finite field is not definable in FPC.

Atserias, Bulatov and Dawar (2007)
Corollary

Solvability of systems of linear equations over any fixed finite field is not definable in FPC.

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Recall: A linear system $Ax = b$ over a field $k$ is solvable if and only if the matrices $A$ and $(A | b)$ have the same rank over $k$. 
FPC — more non-definability results

Corollary

Solvability of systems of linear equations over any fixed finite field is not definable in FPC.

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Recall: A linear system $Ax = b$ over a field $k$ is solvable if and only if the matrices $A$ and $(A \mid b)$ have the same rank over $k$

Corollary

Matrix rank over finite fields is not definable in FPC.
Which matrix properties can be defined in FPC?
Which matrix properties *can* be defined in FPC?

1. **Characteristic polynomial** and **determinant** of a square matrix over $\mathbb{Z}$, $\mathbb{Q}$ and any finite field.

Dawar, H., Grohe, Laubner (2009)
Which matrix properties can be defined in FPC?

1. Characteristic polynomial and determinant of a square matrix over \( \mathbb{Z}, \mathbb{Q} \) and any finite field.
2. The inverse to any invertible square matrix over \( \mathbb{Z}, \mathbb{Q} \) and any finite field.

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4. **Minimal polynomial** of a square matrix over $\mathbb{Q}$ and any finite field.

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Which matrix properties \textit{can} be defined in FPC?

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Fundamental linear-algebraic property over fields that separates FPC from PTIME: rank over finite fields
Which matrix properties can be defined in FPC?

1. Characteristic polynomial and determinant of a square matrix over $\mathbb{Z}$, $\mathbb{Q}$ and any finite field.
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H.-Pakusa (2010)

Fundamental linear-algebraic property over fields that separates FPC from PTIME: rank over finite fields

(Next talk: solvability problems over groups and rings)
Next step: extend fixed-point logic with ability to define matrix rank
Definable matrix relations

Recall: View any $A \subseteq I \times I$ as a matrix over GF(2).
Definable matrix relations

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formula $\varphi(x, y)$

graph $G = (V, E)$
Definable matrix relations

Recall: View any $A \subseteq I \times I$ as a matrix over GF(2).

\[ \varphi(x, y) \rightarrow M^G_{\varphi}: V \to V \]

(formula $\varphi(x, y)$) (graph $G = (V, E)$) (over GF(2))
Definable matrix relations

Recall: View any $A \subseteq I \times I$ as a matrix over GF(2).

$\varphi(x, y)$

$G = (V, E)$

$M^G_{\varphi} : V \rightarrow V$ (over GF(2))
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Recall: View any $A \subseteq I \times I$ as a matrix over GF(2).

formula $\varphi(x, y)$

graph $G = (V, E)$

$M^G_\varphi$: (over GF(2))
Definable matrix relations

Recall: View any \( A \subseteq I \times I \) as a matrix over GF(2).

\[
\varphi(x, y) \quad \rightarrow \quad M^G_{\varphi} : \quad V \rightarrow V \\
\text{(over GF(2))}
\]

Example: \( \varphi(x, y) := E(x, y) \quad \rightarrow \quad M^G_{\varphi} = \text{adjacency matrix of G} \)
Definable matrix relations

Recall: View any \( A \subseteq I \times I \) as a matrix over GF(2).

\[
\begin{align*}
\text{formula} & \quad \varphi(x, y) \\
\text{graph} & \quad G = (V, E) \quad \rightarrow \quad M_G^\varphi : V \rightarrow V \\
\text{(over GF(2))}
\end{align*}
\]

Example: \( \varphi(x, y) := E(x, y) \rightarrow M_G^\varphi = \text{adjacency matrix of } G \)

More generally: formalise matrices over GF(\(p\)), \(p\) prime
Fixed-point logic with rank operators

Variables are typed:

\[ \mathbb{N} \]

\[ G = (V, E) \]

\[ R_1, \ldots, R_k \]

\[ L \]

is a total linear ordering

\[ y \]

is the \( L \)-successor of \( x \)
Fixed-point logic with rank operators

Variables are typed:

\[ R_1, \ldots, R_k. \]

\( L \) is a total linear ordering

\( y \) is the \( L \)-successor of \( x \)

\[ G = (V, E) \]

vertex variables: range over the vertices \( V \)
Fixed-point logic with rank operators

Variables are typed:

- Number variables: range over $\mathbb{N}$
- Vertex variables: range over the vertices $V$

$G = (V, E)$
Fixed-point logic with rank operators

Variables are typed:

\[
\begin{align*}
N &= 0, 1, 2, 3, 4, 5, 6, 7, \ldots \\
\end{align*}
\]

- number variables: range over \( \mathbb{N} \)
- vertex variables: range over the vertices \( V \)

- Bounded quantification over number sort
Fixed-point logic with rank operators

Variables are typed:

- Bounded quantification over number sort
- Extend FP with rules for rank terms: \( \text{rk}_p(x, y) \cdot \varphi \) (\( p \) prime)

\[
G = (V, E)
\]

number variables: range over \( \mathbb{N} \)
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Fixed-point logic with rank operators

Variables are typed:

- Bounded quantification over number sort
- Extend FP with rules for rank terms: \( \text{rk}_p(x, y).\varphi \) (\( p \) prime)

Semantics: \( (\text{rk}_p(x, y).\varphi)^G := \text{rank}(M^G_{\varphi}) \) over GF(\( p \))
Fixed-point logic with rank operators

Variables are typed:

- Bounded quantification over number sort
- Extend FP with rules for rank terms: \( \text{rk}_p(x, y).\varphi \) \( (p \text{ prime}) \)

Semantics: \( (\text{rk}_p(x, y).\varphi)^G := \text{rank}(M^G_\varphi) \) over GF(\(p\))

Logics \( \text{FPR}_p, \text{FPR} \) and similarly \( \text{FOR}_p, \text{FOR} \)
Expressive power of rank logics

For any prime $p$, $\text{FPR}_p$ can express solvability of linear equations over $\text{GF}(p)$.  

Dawar, Grohe, H., Laubner (2009)
Expressive power of rank logics

For any prime $p$, $\text{FPR}_p$ can express solvability of linear equations over $\text{GF}(p^m)$ for any $m$.  

H. (2010)
Expressive power of rank logics

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H. (2010)
Expressive power of rank logics

For any prime $p$, $\text{FPR}_p$ can express solvability of linear equations over $\text{GF}(p^m)$ for any $m$.  

$\begin{bmatrix} t \\ \hline \end{bmatrix}^\top = \begin{bmatrix} \vdots \\ \hline \end{bmatrix}$

over $\text{GF}(p^m)$

Represent each element of $\text{GF}(p^m)$ as an $m$-by-$m$ matrix over $\text{GF}(p)$
Expressive power of rank logics

For any prime $p$, $\text{FPR}_p$ can express solvability of linear equations over $\text{GF}(p^m)$ for any $m$. 

$\begin{array}{c}
\text{over } \text{GF}(p^m) \\
\text{equivalent system over } \text{GF}(p)
\end{array}$

Represent each element of $\text{GF}(p^m)$ as an $m$-by-$m$ matrix over $\text{GF}(p)$
Expressive power of rank logics

For any prime $p$, $\text{FPR}_p$ can express solvability of linear equations over $\text{GF}(p^m)$ for any $m$. 

$\text{FPC} \subset \text{FPR}_p \subseteq \text{PTIME}$.

Represent each element of $\text{GF}(p^m)$ as an $m$-by-$m$ matrix over $\text{GF}(p)$.
Expressive power of rank logics

For any prime $p$, $\text{FPR}_p$ can express solvability of linear equations over $\text{GF}(p^m)$ for any $m$.

H. (2010)

Represent each element of $\text{GF}(p^m)$ as an $m$-by-$m$ matrix over $\text{GF}(p)$

\[ \begin{array}{c}
\text{over } \text{GF}(p^m) \\
= \\
\text{equivalent system over } \text{GF}(p)
\end{array} \]

Corollary

For any prime $p$, $\text{FPC} \subset \text{FPR}_p \subset \text{PTIME}$.

(we can simulate counting by expressing rank of diagonal matrices)
CFI graphs revisited

Non-isomorphic CFI graphs can be distinguished by a sentence of FOR$_2$.

Dawar, Grohe, H., Laubner (2009)
CFI graphs revisited

Non-isomorphic CFI graphs can be distinguished by a sentence of FOR$_2$.

Recall: FPC does not capture PTIME on graphs of bounded colour-class size \( \mapsto \) not even size 4

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CFI graphs revisited

Non-isomorphic CFI graphs can be distinguished by a sentence of \( \text{FOR}_2 \).

\[ \text{Dawar, Grohe, H., Laubner (2009)} \]

\textbf{Recall: } FPC does not capture \text{PTIME} on graphs of bounded colour-class size \( \rightarrow \) not even size 4

\textbf{Isomorphism} of graphs of colour class size 4 can be expressed in \( \text{FOR}_2 \).

\[ \text{Dawar, H. (2011)} \]
Pebble games for rank logics & the Weisfeiler-Lehman method
Proving non-definability in FPR

Recall: Proofs of inexpressibility in FPC are generally formulated using a game method.
Proving non-definability in FPR\textsubscript{p}

Recall: Proofs of inexpressibility in FPC are generally formulated using a game method.

Our wish list:
Proving non-definability in $\text{FPR}_p$

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A pebble game for finite-variable rank logics for which...
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1. we can decide who wins the game in polynomial time, and
Proving non-definability in FPR$_p$

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Proving non-definability in FPR<sub>p</sub>

\[ R^k_p \] — first-order logic with variables \( x_1, ..., x_k \) and rank quantifiers of the form \( \text{rk}^i_p(x, y) \cdot (\varphi) \)
Proving non-definability in FPR\textsubscript{p}

\[ R^k_p \] — first-order logic with variables \( x_1, \ldots, x_k \) and \textbf{rank quantifiers} of the form \( \text{rk}^i_p(x, y) \cdot (\varphi) \)

1. Every formula of FPR\textsubscript{p} is invariant under \( R^k_p \) - equivalence, for some \( k \).
Proving non-definability in FPR\(_p\)

\(R^k_p\) — first-order logic with variables \(x_1, ..., x_k\) and rank quantifiers of the form \(r\mathbb{K}^i_p(x, y) \cdot (\varphi)\)

1. Every formula of FPR\(_p\) is invariant under \(R^k_p\) - equivalence, for some \(k\).
2. \(R^k_p\)-equivalence can be characterised by a \(k\)-pebble matrix-rank game (over GF\((p)\))
Proving non-definability in $\text{FPR}_p$

$R_p^k$ — first-order logic with variables $x_1, ..., x_k$ and rank quantifiers of the form $\text{rk}^\geq_i(x, y) \cdot (\varphi)$

1. Every formula of $\text{FPR}_p$ is invariant under $R_p^k$-equivalence, for some $k$.
2. $R_p^k$-equivalence can be characterised by a $k$-pebble matrix-rank game (over $\text{GF}(p)$)

$G$ and $H$ agree on all sentences of $k$-variable rank logic over $\text{GF}(p)$ iff Duplicator has a winning strategy in the $k$-pebble matrix-rank game on $G$ and $H$
Matrix-rank game over GF(\(p\))
Matrix-rank game over GF($p$)

Game played on finite graphs $G$ and $H$
Matrix-rank game over $\text{GF}(\mathbb{F})$

**Game played on finite graphs $G$ and $H$**

- Protocol based on partitioning each game board into disjoint $\{0,1\}$-matrices ("partition matrices").
Matrix-rank game over $\text{GF}(p)$

Game played on finite graphs $G$ and $H$

• Protocol based on partitioning each game board into disjoint $\{0,1\}$-matrices ("partition matrices").

• **Algebraic game rules**: At each round, Duplicator has to ensure that every linear combination of partition matrices over $G$ has the same $\text{GF}(p)$-rank as the corresponding linear combination over $H$. 
Matrix-rank game over GF($p$)

**Problem:** Not known if we can decide in polynomial time which player wins the game.

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Strengthening the game rules

Two tuples \((A_1, A_2, ..., A_m)\) and \((B_1, B_2, ..., B_m)\) of \(n\)-by-\(n\) matrices over a field \(k\) are simultaneously similar if there is an invertible \(S\) such that \(S A_i S^{-1} = B_i\) for all \(i\).
Strengthening the game rules

Two tuples \((A_1, A_2, ..., A_m)\) and \((B_1, B_2, ..., B_m)\) of \(n\)-by-\(n\) matrices over a field \(k\) are \textbf{simultaneously similar} if there is an invertible \(S\) such that \(S A_i S^{-1} = B_i\) for all \(i\).

There is a deterministic algorithm that, given two \(m\)-tuples \(A\) and \(B\) of \(n\)-by-\(n\) matrices over a finite field \(GF(q)\), determines in time \(\text{poly}(n, m, q)\) whether \(A\) and \(B\) are simultaneously similar.

\text{Chistov, Karpinsky and Ivanyov (1997)}
Game based on invertible linear maps

Invertible-map game on $G$ and $H$ over $\text{GF}(p)$:

- Protocol based on partitioning each game board into disjoint $\{0,1\}$-matrices (“partition matrices”).
Game based on invertible linear maps

Invertible-map game on $G$ and $H$ over GF($p$):

- Protocol based on partitioning each game board into disjoint \{0,1\}-matrices ("partition matrices").

- New game rule: At each round, Duplicator has to ensure that the two tuples of partition matrices (over $G$ and $H$) are \textit{simultaneously similar} over GF($p$).
Game based on invertible linear maps

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**Facts:**
Game based on invertible linear maps

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Facts:

- We can decide who wins this game in PTIME.
Game based on invertible linear maps

Invertible-map game on $G$ and $H$ over $\text{GF}(p)$:

- Protocol based on partitioning each game board into disjoint $\{0,1\}$-matrices ("partition matrices").

- **New game rule:** At each round, Duplicator has to ensure that the two tuples of partition matrices (over $G$ and $H$) are *simultaneously similar* over $\text{GF}(p)$.

**Facts:**

- We can decide who wins this game in PTIME.

- **Refines** $R^k_p$-equivalence: If Duplicator wins the $k$-pebble invertible-map game on $G$ and $H$ then she also wins the $k$-pebble matrix rank game on $G$ and $H$. 
Connection with stable colouring

Recall:

Our wish list:

A pebble game for finite-variable rank logics for which...

1. we can decide who wins the game in polynomial time, and

2. there is a corresponding “stable colouring algorithm”, like for the counting game on graphs.
Weisfeiler-Lehman refinement

**Input:** Graph \( G = (V, E) \)

**Output:** Equivalence relation \( \approx \) on \( V \).
Weisfeiler-Lehman refinement

Input: Graph $G = (V, E)$
Output: Equivalence relation $\approx$ on $V$.

"colour refinement" or "stable colouring"
Weisfeiler-Lehman refinement

Input: Graph $G = (V, E)$
Output: Equivalence relation $\sim$ on $V$.

Inductively define: $\sim_0 \supseteq \sim_1 \supseteq \ldots \supseteq \sim_m = \sim_{m+1} =: \approx$
Weisfeiler-Lehman refinement

Input:  \( \text{Graph } G = (V, E) \)
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Inductively define: \( \sim_0 \supseteq \sim_1 \supseteq \ldots \supseteq \sim_m = \sim_{m+1} =: \approx \)

Initial: \( u \sim_0 v \) iff \( \deg(u) = \deg(v) \)
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**Initial:** $u \sim_0 v$ iff $\deg(u) = \deg(v)$

**Refine:** $u \sim_{i+1} v$ iff $u \sim_i v$ and for all $\alpha \in V/\sim_i$:

$$\|N(u) \cap \alpha\| = \|N(v) \cap \alpha\|$$
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$\alpha = \{w \mid \deg(w) = 2\}$
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Weisfeiler-Lehman algorithm for GI

Input: Graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$
Output: “isomorphic” or “not isomorphic”
Weisfeiler-Lehman algorithm for GI

Input: Graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$
Output: "isomorphic" or "not isomorphic"

1. Compute the WL refinement $\cong$ on $G \cup H$
2. Output "not isomorphic" if there is some $\alpha \in G \cup H / \cong$ such that $\|\alpha \cap V_G\| \neq \|\alpha \cap V_H\|$; else "isomorphic".
Weisfeiler-Lehman algorithm for GI

**Input:** Graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$

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Some facts:
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Some facts:
1. WL runs in time $O(n^2 \log(n))$
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Some facts:
1. WL runs in time $O(n^2 \log(n))$
2. WL is correct almost surely Babai, Erdös and Selkow (1980)
Weisfeiler-Lehman algorithm for GI

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Some facts:
1. WL runs in time $O(n^2 \log(n))$
2. WL is correct almost surely \cite{Babai1980}
3. WL fails on non-isomorphic regular graphs
\textbf{$k$-dimensional WL$^*$ refinement}

One-element extensions in $G = (V, E)$

For $\alpha \subseteq V^k$, a k-tuple $\tilde{u} \in V^k$ and $0 \leq i < k$, let:

$$\Gamma_i(\tilde{u}, \alpha) := \{ w \in V \mid \tilde{u}^w_i \in \alpha \}$$
**$k$-dimensional WL$^*$ refinement**

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$$\Gamma_i(\vec{u}, \alpha) := \{ w \in V \mid \vec{u}_{<i}^w \in \alpha \}$$

**Example:** Let $k = 3$ and $\alpha := \{(x, y, z) \in V^3 \mid (x, y, z) = \triangle \}$
**k-dimensional WL* refinement**

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Graph:

- $w$, $a$, $b$, $v$, $u$

Equations:

$$\Gamma_0(uvw, \alpha) = \{a, b\}$$

$$\Gamma_1(uvw, \alpha) = \emptyset$$
$k$-dimensional WL* refinement

**Input:** Graph $G = (V, E)$

**Output:** Equivalence relation $\cong$ on $V^k$. 
**k-dimensional WL* refinement**

**Input:** Graph $G = (V, E)$

**Output:** Equivalence relation $\approx$ on $V^k$.

**Initial:** $\vec{u} \sim_0 \vec{v}$ iff $\text{atp}_G(\vec{u}) = \text{atp}_G(\vec{v})$
**k-dimensional WL\(^*\) refinement**

**Input:** Graph G = (V, E)

**Output:** Equivalence relation \(\sim\) on \(V^k\).

**Initial:** \(\vec{u} \sim_0 \vec{v}\) iff \(\text{atp}_G(\vec{u}) = \text{atp}_G(\vec{v})\)

**Refine:** \(\vec{u} \sim_{m+1} \vec{v}\) iff \(\vec{u} \sim_m \vec{v}\) and for all \(0 \leq i < k\) there is a bijection \(f : V \rightarrow V\) s.t.

\[
f : \Gamma_i(\vec{u}, \alpha) \mapsto \Gamma_i(\vec{v}, \alpha)
\]

for all \(\alpha \in V^k / \sim_m\)
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**Input:** Graph $G = (V, E)$

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---

# k-dimensional WL* refinement

**Input:** Graph $G = (V, E)$

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$$\Gamma_i(\vec{u}, \alpha) := \{w \in V \mid \vec{u}^w_i \in \alpha\}$$

$$f : \Gamma_i(\vec{u}, \alpha) \mapsto \Gamma_i(\vec{v}, \alpha)$$

for all $\alpha \in V^k / \sim_m$

**Theorem:** $\vec{u} \approx \vec{v}$ iff they agree on all $C^k$-formulas in $G$. 
\textit{k-dimensional WL* algorithm for GI}

As before: compute \textit{k-dimensional WL* refinement and compare across the two graphs.}

\textbf{PTIME for fixed \textit{k}:} \textit{k-dim WL*} runs in time $O(n^{k+1} \log(n))$. 
$k$-dimensional WL* algorithm for GI

As before: compute $k$-dimensional WL* refinement and compare across the two graphs.

PTIME for fixed $k$: $k$-dim WL* runs in time $O(n^{k+1} \log(n))$.

There exists a sequence of pairs $\{(G_n, H_n)\}_n$ of non-isomorphic graphs for which it holds that:

- $G_n$ and $H_n$ have $O(n)$ vertices but
- $G_n$ and $H_n$ are not distinguished by the $n$-dim WL* algorithm.

Cai, Fürer and Immerman (1992)
Refinement by invertible linear maps

Two-element extensions in $G = (V, E)$

For $\alpha \subseteq V^k$, a $k$-tuple $\vec{u} \in V^k$ and $0 \leq i \neq j < k$, let:

$$\Gamma_{ij}(\vec{u}, \alpha) := \{(a, b) \in V \times V \mid \vec{u}_{\frac{a}{i}}^{\frac{b}{j}} \in \alpha\} \subseteq V \times V$$
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Example: Let $k = 3$ and $\alpha := \{(x, y, z) \in V^3 \mid (x, y, z) = \triangle \}$

Diagram:

![Diagram](image)
Refinement by invertible linear maps

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Example: Let $k = 3$ and $\alpha := \{(x, y, z) \in V^3 | (x, y, z) = \square\}$
$k$-dimensional IM refinement over $\text{GF}(p)$

**Input:** Graph $G = (V, E)$

**Output:** Equivalence relation $\sim$ on $V^k$. 
$k$-dimensional IM refinement over $\text{GF}(p)$

Input: Graph $G = (V, E)$
Output: Equivalence relation $\sim$ on $V^k$.

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**Refine:** $\vec{u} \sim_{m+1} \vec{v}$ iff $\vec{u} \sim_m \vec{v}$ and for all $0 \leq i \neq j < k$

there is $S \in \text{GL}_V(\text{GF}(p))$ s.t.

$$S \cdot \Gamma_{ij}(\vec{u}, \alpha) \cdot S^{-1} = \Gamma_{ij}(\vec{v}, \alpha)$$

for all $\alpha \in V^k / \sim_m$
$k$-dimensional $\text{IM}_p$ algorithm for GI

Similar to WL: compute $k$-dimensional IM refinement and compare across the two graphs (here over GF($p$))
**k-dimensional IM<sub>p</sub> algorithm for GI**

Similar to WL: compute <i>k</i>-dimensional IM refinement and compare across the two graphs (here over GF(<i>p</i>))

- For each <i>k</i>, <i>k</i>-dim IM<sub>p</sub> runs in **polynomial time** for all <i>p</i>.
- **Refinement**: <i>k</i>-dim WL* ⊇ (k+1)-dim IM<sub>p</sub> ⊇ (k+2)-dim IM<sub>p</sub>
**k-dimensional IM$_p$ algorithm for GI**

Similar to WL: compute $k$-dimensional IM refinement and compare across the two graphs (here over GF($p$))

- For each $k$, $k$-dim IM$_p$ runs in polynomial time for all $p$.
- **Refinement**: $k$-dim WL$^*$ $\supseteq$ $(k+1)$-dim IM$_p$ $\supseteq$ $(k+2)$-dim IM$_p$

For each $k$ and prime $p$, there is a pair of non-isomorphic graphs that can be distinguished by 3-dim IM$_p$ but not by $k$-dim WL$^*$.  

Dawar and H. (2012)
**k-dimensional IM\(_p\) algorithm for GI**

Similar to WL: compute \(k\)-dimensional IM refinement and compare across the two graphs (here over GF\((p)\))

- For each \(k\), \(k\)-dim IM\(_p\) runs in **polynomial time** for all \(p\).
- **Refinement**: \(k\)-dim WL\(^*\) \(\supseteq (k+1)\)-dim IM\(_p\) \(\supseteq (k+2)\)-dim IM\(_p\)

For each \(k\) and prime \(p\), there is a pair of non-isomorphic graphs that can be distinguished by 3-dim IM\(_p\) but not by \(k\)-dim WL\(^*\).

Dawar and H. (2012)

For each \(k\) and distinct primes \(p\) and \(q\), there is a pair of non-isomorphic graphs that can be distinguished by 3-dim IM\(_p\) but not by \(k\)-dim IM\(_q\).  

H. (2010)
$k$-dimensional IM$_p$ more generally

Consider the invertible-map algorithm for larger matrices (higher arity) and finite sets of primes.

Can we give instances where the general algorithm fails to express graph isomorphism?
Some open problems
Problem 1: Separate $\text{FOR}_p$ and $\text{FOR}_q$ over empty signatures

For formula $\varphi(x, y)$, integer $n$ and prime $p$, let $r^p_\varphi(n)$ denote the $\text{GF}(p)$-rank of the matrix defined by $\varphi(x, y)$ over an $n$-element set.
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**Polynomial-rank conjecture**
For each \(\varphi(x, y)\) and each prime \(p\), there are unary polynomials \(f_0, \ldots, f_{p-1}\) such that \(r^p_\varphi(n) = f_i(n)\) for all (sufficiently large) \(n\) congruent to \(i\) modulo \(p\).
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H. and Laubner (2010)
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Kirsten (2012)
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Problem 2: Give capturing results for FPR on natural classes of graphs

Consider classes on which we know that FPC does not capture PTIME:

• graphs of bounded degree
• graphs of bounded colour-class size
Further questions

- Can FPR express matching in arbitrary graphs?
- Does the "simultaneous similarity game" correspond to a natural logic?

More open problems to come in the next talk!